AN OVERVIEW OF RAMANUJAN'S NOTEBOOKS

Bruce C. Berndt

Generally acknowledged as India's greatest mathematician, Srinivasa Ramanujan was born on 22 December 1887 in Erode, located in the southern Indian State of Tamil Nadu. He began to focus on mathematics at an early age, and, at the age of about fifteen, borrowed a copy of G. S. Carr's *Synopsis of Pure and Applied Mathematics* [29], which served as his primary source for learning mathematics. Carr was a tutor and compiled this compendium of approximately 4000–5000 results (with very few proofs) to facilitate his tutoring. At the age of sixteen, Ramanujan entered the Government College in Kumbakonam, where he lived for most of his life. By that time, Ramanujan was solely devoted to mathematics and consequently failed his examinations at the end of the first year. He therefore lost his scholarship and, because his family was poor, was forced to terminate his formal education.

At about the time Ramanujan entered college, he began to record his mathematical discoveries in notebooks, although the entries on magic squares in Chapter 1 probably emanate from his school days. Living in poverty with no means of financial support, suffering at times from serious illnesses, and working in isolation, Ramanujan devoted all of his efforts to mathematics and continued to record his discoveries without proofs in notebooks for the next six years.

In 1910 Ramanujan moved to Madras where he obtained temporary financial assistance from R. Ramachandra Rao, and subsequently became a clerk in the Madras Port Trust Office. Several people offered support, and Ramanujan was persuaded to write English mathematicians about his mathematical discoveries. One, G. H. Hardy, responded encouragingly and invited Ramanujan to come to Cambridge to develop his mathematical gifts. After some reluctance, Ramanujan boarded a passenger ship for England on 17 March 1914. At about this time, Ramanujan evidently stopped recording his theorems in notebooks, although a few entries in his third notebook were undoubtedly recorded in England. That Ramanujan no longer concentrated on logging entries in his notebooks is evident from two letters that he wrote friends in Madras during his first year in England [25]. In a letter of 13 November 1914 to his friend R. Krishna Rao, Ramanujan confided, "I have changed my plan of publishing my results. I am not going to publish any of the old results in my notebooks till the war is over." And in a letter of 7 January 1915 to S. M. Subramanian, Ramanujan admitted, "I am doing my work very slowly. My notebook is sleeping in a corner for these four or five months. I am publishing only my present researches as I have not yet proved the results in my notebooks rigorously."

On 24 March 1915, near the end of his first winter in Cambridge, Ramanujan wrote his friend E. Vinayaka Row in Madras, "I was not well till the beginning of this term owing to the weather and consequently I couldn't publish any thing

Typeset by $\mathcal{AMS}$-TEX

for about 5 months." By the end of his third year in England, Ramanujan was critically ill, and, for the next two years, he was confined to sanitariums and nursing homes. After World War I ended in 1919, Ramanujan returned home, but his health continued to deteriorate, and on 26 April 1920 Ramanujan died at the age of 32.

Doctors in both England and India had difficulty diagnosing his illness; tuberculosis and a severe vitamin deficiency were perhaps the two most frequently suggested causes. However, D. A. B. Young [74] has recently made a careful examination of all the records and symptoms of Ramanujan's illness and has convincingly concluded that Ramanujan suffered from hepatic amoebiasis (parasitic infection of the liver). Not only do all of Ramanujan's symptons suggest this disease, but Ramanujan's medical history in India also favors this diagnosis. Amoebiasis is a protozoal infection of the large intestine that gives rise to dysentery. The disease is widespread in India, particularly around coastal cities such as Madras. In 1906 Ramanujan left home to attend Pachaiyappa's College in Madras, where he contracted a severe case of dysentery and had to return home for three months. Unless adequately treated, the infection is permanent, although the patient may go for long periods without exhibiting any symptons. Relapses occur when the host–parasite relationship is disturbed, which likely happened in 1909 when Ramanujan again became acutely ill. This illness is very difficult to diagnose, but once diagnosed, it can be readily cured.

Our description of Ramanujan's life has been necessarily brief. For several years, the standard sources about Ramanujan's life have been the obituaries of P. V. Seshu Aiyar, R. Ramachandra Rao, and Hardy, found in Ramanujan's *Collected Papers* [54], and Chapter 1 of Hardy's book [38]. By far, the most comprehensive biography of Ramanujan has been written by R. Kanigel [40]. The letters from and to Ramanujan are also a source of both mathematical and personal information about Ramanujan, and most of the extant letters have recently been compiled with commentary by R. A. Rankin and the author [25].

After Ramanujan died, Hardy strongly urged that Ramanujan's notebooks be edited and published. By "editing," Hardy meant that each claim made by Ramanujan in his notebooks should be examined. If a theorem is known, sources providing proofs should be cited; if an entry is not known, then an attempt should be made to prove it. Ramanujan, in fact, had left his first notebook with Hardy when he returned to India in 1919, and in 1923 Hardy wrote a paper [36], [37, pp. 505–516] about a chapter on hypergeometric series found in the first notebook. In this paper, Hardy pointed out that Ramanujan had independently discovered most of the important classical results in the subject while also discovering several new theorems as well. It should be remarked here that the second notebook contains two expanded chapters on hypergeometric series, and that further results on hypergeometric series, most of them new, can be found at other scattered places in the second and third notebooks.

Hardy sent the first notebook to the University of Madras where Ramanujan's other notebooks and papers were being preserved. Later, in 1925, T. A. Satagopan at the University of Madras made a handwritten copy of the first notebook which was mailed to Hardy. Plans were being made to publish Ramanujan's collected papers and, possibly, his notebooks and other manuscripts. Thus, on 30 August 1923, the Registrar at the University of Madras, Francis Dewsbury, sent the second notebook (in four parts) to Hardy, and in 1925 further papers were sent to him. The original notebooks were returned to Madras, but the remaining papers evidently were not. It transpired that Ramanujan's *Collected Papers* [54] were published in 1927, but his notebooks and other manuscripts were not published.

Sometime in the late 1920s, G. N. Watson and B. M. Wilson began the task of editing Ramanujan's notebooks. The second notebook, being a revised, enlarged edition of the first, was their primary focus. Wilson was assigned Chapters 2–14, and Watson was to examine Chapters 15–21. Wilson devoted his efforts to this task until 1935, when he died from an infection at the early age of 38. Watson wrote over 30 papers inspired by the notebooks before his interest evidently waned in the late 1930s. Thus, the project was never completed.

It was not until 1957 that the notebooks were made available to the public when the Tata Institute of Fundamental Research in Bombay published a photocopy edition [55], but no editing was undertaken. The first notebook was published in volume 1, and volume 2 comprises the second and third notebooks.

While residing for a year at the Institute for Advanced Study in Princeton, on a cold winter day in early February, 1974, I was reading two papers by Emil Grosswald [34], [35] in which some formulas from the notebooks were proved. I observed that I could prove these formulas by using a theorem I had proved two years earlier on the modular transformations of a large class of functions including the Dedekind eta–function. I was naturally curious to determine if there were other formulas in the notebooks which I could prove employing my methods. Fortunately, the library at Princeton University has a copy of the Tata Institute's edition, and, indeed, I found a few more formulas of the same sort which I could prove. Eventually I wrote two long papers [7], [8] providing proofs of several formulas from the notebooks and many others of a kindred nature.

All of the aforementioned formulas of Ramanujan can be found in Chapter 14 of his second notebook. However, there were many beautiful formulas involving infinite series in Chapter 14 that I could not prove. So, after the spring semester at the University of Illinois ended in May, 1977, I decided to attempt to find proofs for all 87 formulas in Chapter 14. After working on this project for close to a year, George Andrews, in a visit at the University of Illinois, informed me that Watson and Wilson's efforts in editing the notebooks were preserved in Trinity College Library at Cambridge. The librarian kindly sent me a copy of their notes, which have been enormously helpful. Especially helpful have been Watson's notes on the chapters on modular equations. Thus, since May, 1977, I have devoted all of my research efforts to editing Ramanujan's notebooks. This work, accomplished with the help of several mathematicians, is contained in [9], [10], [11], [12], and [13].

A few remarks should be offered about other unpublished manuscripts left by Ramanujan. The most celebrated and important manuscript, Ramanujan's "lost notebook," was rediscovered by Andrews in 1976 at Trinity College Library, Cambridge. An introduction to the "lost notebook" can be found in Andrews' article [1]. The history of this "notebook," actually a sheaf of 138 pages, is unclear. It probably was sent in 1923 from Madras to Hardy, who kept it in his possession until possibly the late 1930s or early 1940s when he passed it to Watson, who, by that time, no longer had the passionate interest in Ramanujan's work that he had earlier. The "notebook" was found in Watson's papers after his death in 1965. An obituary of Watson was written by R. A. Rankin [57], who, in December, 1968, sent Ramanujan's manuscript along with several papers of Watson to Trinity College Library. The "lost notebook" was published along with other manuscripts and letters by Ramanujan in 1988 [55].

As indicated above, Ramanujan left three notebooks. The first notebook, totalling 351 pages, contains 16 chapters of loosely organized material with the remainder unorganized. In the organized part, which ends on page 263 in the pagination of [55], Ramanujan wrote on only one side of the paper. Shortly thereafter, Ramanujan began to write on both sides of the page and then returned to the unused reverse sides to record additional material, so that only about 20 of the 351 pages are actually blank.

The second notebook is a revised enlargement of the first and was probably composed during the nine months that Ramanujan held a scholarship at the University of Madras prior to his departure for England. This notebook contains 21 chapters, comprising 256 pages, followed by 100 pages of miscellaneous material.

The short third notebook contains 33 pages of unorganized entries.

Hardy estimated that the notebooks contain approximately 3000–4000 results. This estimate appears to be correct; in preparing *Ramanujan's Notebooks, Parts I–V* [9]–[13], we counted 3254 results, although we emphasize that different people will tally such a count in different ways. Each chapter contains approximately 50–150 entries.

Hardy estimated that 2/3 of Ramanujan's work in India was rediscovery. This estimate is definitely too high, for at least 1/2 of the material was new when our editing commenced in 1977.

As indicated above, the notebooks contain almost no proofs. Perhaps there are about 10–20 results for which Ramanujan sketches a proof, often with only one sentence. There are several reasons for the absence of proofs.

1. Ramanujan was probably influenced by the style of Carr's book [29].

2. Like most Indian students in his time, Ramanujan worked primarily on a slate. Paper was very expensive. Thus, after rubbing out his proofs with his sleeve, Ramanujan recorded only the final results in his notebooks.

3. Ramanujan never intended that his notebooks be made available to the mathematical public. They were his personal compilation of what he had discovered. If someone had asked him how to prove a particular result in the notebooks, undoubtedly Ramanujan could supply a proof.

Many have speculated about Ramanujan's methods. Indeed, for many parts of Ramanujan's mathematics, it is difficult to make intelligent guesses about how he argued. In other areas, we can be fairly certain about the nature of many of Ramanujan's arguments, although we may not know the precise details. It should be emphasized, however, that Ramanujan doubtless thought like any other mathematician; he just thought with more insight than most of us. Although Ramanujan, and others among us, may attribute our revelations to divine inspiration, it does not help us to understand Ramanujan's discoveries by claiming that they were obtained by intuition, the inspiration of Goddess Namagiri, or some other mystical means.

Since Ramanujan's notebooks were only intended for his personal use, we might surmise that they contain several errors. Of course, there are occasional misprints. However, perhaps surprisingly, there are very few serious mistakes. Because Ramanujan had little formal training, his proofs were undoubtedly not rigorous in many instances. Despite this, Ramanujan was keenly aware of when his unrigorous thinking yielded correct results, and when it did not. Most of Ramanujan's mistakes arise from his claims in analytic number theory, where his unrigorous methods led him astray. In particular, Ramanujan thought that his approximations and asymptotic expansions were considerably more accurate than were warranted. In [12], these shortcomings are discussed in detail. However, a warning should be given to those examining the notebooks for the first time. It is easy to conclude that many formulas are incorrect. Ramanujan often recorded results in unconventional manners, but, properly interpreted, almost always Ramanujan is correct.

Although Ramanujan is primarily known to the mathematical community as a number theorist, only a small portion of the material in the notebooks is devoted to number theory. Most of the contents come under the purview of classical analysis. However, numerous results, e.g., the several hundred theorems on theta functions and modular equations, are at the interface of analysis and number theory. Opening the notebooks, one will likely focus on some infinite series. Infinite series were undoubtedly Ramanujan's first love; perhaps only Euler possessed Ramanujan's talents in working with infinite series.

Next, we briefly discuss several topics examined by Ramanujan in his notebooks. In this brief account, we cannot give a complete description of Ramanujan's important contributions to any of these areas, and indeed many topics will not be discussed at all. We have moderately emphasized Ramanujan's claims in the unorganized portions of the notebooks. To keep the number of references to a manageable number, we ask readers to consult our books [9]–[13], where additional history and fuller lists of references are provided.

1. Elementary Mathematics

Many of Ramanujan's discoveries can be appreciated by those with only a knowledge of high school algebra. Chapter 2 in the second notebook, the unorganized portions of the second and third notebooks [12, Chapter 22], and the problems that Ramanujan submitted to the *Journal of the Indian Mathematical Society* [54] are excellent sources for these gems.

Those familiar with the famous taxicab number 1729 might correctly surmise that Ramanujan enjoyed finding equal sums of powers. Thus, for example, if $a + b + c = 0$ [12, p. 96],

$$2(ab + ac + bc)^4 = a^4(b - c)^4 + b^4(c - a)^4 + c^4(a - b)^4.$$

In fact, in his third notebook [55, p. 385], Ramanujan recorded similar formulas for $2(ab + ac + bc)^{2n}$, for $n = 1$–4, and wrote "and so on" to indicate that he possessed a general procedure for finding such formulas. We think S. Bhargava found Ramanujan's secret here, and we refer the reader to his elegant theorem in [12, p. 97].

One of Ramanujan's most remarkable formulas is the following polynomial identity. Let

$$\begin{aligned} F_{2m}(a, b, c, d) =& (a + b + c)^{2m} + (b + c + d)^{2m} - (c + d + a)^{2m} \\ &- (d + a + b)^{2m} + (a - d)^{2m} - (b - c)^{2m}. \end{aligned}$$

Then [12, p. 102]

$$64F_6(a, b, c, d)F_{10}(a, b, c, d) = 45F_8^2(a, b, c, d),$$

first proved by Bhargava and the author [12, pp. 102–104].

Did you know that [12, pp. 19, 39]

$$2\sin(\pi/18) = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \cdots}}}},$$

where the sequence of signs $-, +, +, \ldots$ has period 3, or that

$$(\cos 40^o)^{1/3} + (\cos 80^o)^{1/3} - (\cos 20^o)^{1/3} = \sqrt[3]{\tfrac{3}{2}(\sqrt[3]{9} - 2)}?$$

Ramanujan was fond of stating intriguing formulas such as those above, but as in most instances, they are special cases of more general theorems that he established.

2. Number Theory

We cite only one theorem in the notebooks from the theory of numbers, as reformulated by K. S. Williams [12, p. 71].

Theorem. *Let a, b, A, and B denote positive integers satisfying the conditions*

$$(a, b) = 1 = (A, B), \qquad ab \neq \textit{square}.$$

Suppose also that every prime $p \equiv B \pmod{A}$ with $(p, 2ab) = 1$ is expressible in the form $ax^2 - by^2$ for some integers x and y. Then every prime q such that $q \equiv -B \pmod{A}$ and $(q, 2ab) = 1$ is expressible in the form $bX^2 - aY^2$ for some integers X and Y.

As an example, let $a = 1, b = 7, A = 28$, and $B = 9$. If p is a prime such that $p \equiv 9 \pmod{28}$, there exist integers x and y such that $p = x^2 - 7y^2$ [42, Table III]. For example, $37 = 10^2 - 7 \cdot 3^2$. Then by the theorem of Ramanujan and Williams, if q is a prime such that $q \equiv -9 \pmod{28}$, then there exist integers X and Y such that $q = 7X^2 - Y^2$. For example, $19 = 7 \cdot 2^2 - 3^2$.

Many of Ramanujan's theorems on theta functions and modular equations have applications to number theory. In particular, see Sections 10 and 12.

We also remark that a forerunner of the Hardy–Ramanujan "circle method" can be found in the notebooks. Ramanujan attempted (unrigorously) to use the influence of several singularities of the generating function in order to obtain an asymptotic formula [12, pp. 62–66]. Exploiting the function's behavior around many singularities is the crux of the "circle method."

3. Infinite Series

It seems appropriate to begin this section with one of the formulas that Grosswald [34], [35] proved and which fostered our addiction to Ramanujan's formulas. Let $\zeta(s)$ denote the Riemann zeta–function. If $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and n is any nonzero integer, then

$$\begin{aligned} &\alpha^{-n}\left(\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k} - 1}\right) \\ =&(-\beta)^{-n}\left(\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k} - 1}\right) \\ &- 2^{2n}\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^k, \end{aligned} \tag{3.1}$$

where $B_j, j \geq 0$, denotes the jth Bernoulli number. For a discussion of the many proofs of (3.1), see [10, p. 276]. Curiously, in his first notebook, Ramanujan stated a more general result [12, pp. 429, 430], which he did not record in his second notebook. If $\alpha = \beta = \pi$ and n is odd and positive, (3.1) reduces to

$$\zeta(2n+1) = 2^{2n}\pi^{2n+1}\sum_{k=0}^{n+1}(-1)^{k+1}\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!} - 2\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\pi k}-1},$$

which was first discovered by Lerch [43] in 1901. Thus, $\zeta(2n+1)$ equals a rational multiple of π^{2n+1} plus a very rapidly convergent series, i.e., $\zeta(2n+1)/\pi^{2n+1}$ is "almost" rational.

One of Riemann's proofs of the functional equation of $\zeta(s)$ employs the theta transformation formula, which is easily established via the Poisson summation formula, and which was also discovered by Ramanujan [10, p. 253], [11, p. 43]. Quite remarkably, Ramanujan went one step further and found a transformation for doubly exponential series, which he stated in a peculiar fashion in the unorganized portion of his second notebook [55, p. 279]. We offer a more precise version, established in a paper with J. L. Hafner [24]. Let $n, \alpha, \beta > 0$ with $\alpha\beta = 2\pi$. Then

$$\alpha\sum_{k=0}^{\infty}e^{-ne^{k\alpha}} = \alpha\left(\frac{1}{2} + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}n^k}{k!(e^{k\alpha}-1)}\right) - \gamma - \log n + 2\sum_{k=1}^{\infty}\varphi(k\beta),$$

where γ denotes Euler's constant and

$$\varphi(\beta) = \frac{1}{\beta}\mathrm{Im}\left\{n^{-i\beta}\Gamma(i\beta+1)\right\}.$$

This can also be proved by means of the Poisson summation formula, but the proof is considerably more intricate than that for the theta transformation formula.

We give one more example. Let $n, \alpha, \beta > 0$ with $\alpha\beta = 2\pi$. Let

$$\psi(n) = \int_0^{\infty}\varphi(x)\cos(nx)dx$$

(for a suitable class of functions φ). Then on page 312 in his second notebook, Ramanujan claims that

$$\frac{\alpha}{2}\sum_{n=1}^{\infty}\frac{\mu(n)\varphi(\alpha/n)}{n} = \sum_{n=1}^{\infty}\frac{\mu(n)\varphi(\beta/n)}{n}, \tag{3.2}$$

where $\mu(n)$ denotes the Möbius function. The reader need not be told that this "Möbius analogue" of the Poisson summation formula is astounding! Well, (3.2) is too astounding; it is false! However, (3.2) can be corrected by adding a series arising from the complex zeros of $\zeta(s)$. See [13, Chapter 35, Entries 35–37] for further discussion and examples.

The reader will have noticed that the three examples we have chosen have a symmetry in α and β. Indeed, Ramanujan derived many formulas of this type. He also originally summed many infinite series in closed form, derived many beautiful partial fraction expansions, and discovered several summation formulas akin to the

the Abel–Plana summation formula, among his hundreds of beautiful discoveries in the realm of infinite series.

4. Integrals

Although Ramanujan evidently devoted considerably more effort to infinite series than to integrals, several integrals bear his name or are important in current research. Chapter 13 in his second notebook [10] and Chapter 28 in [12], which is a compilation of his results on integrals scattered among the unorganized portions of the second and third notebooks, contain most of the notebooks' entries on integrals, while several of his papers [54] focus on integrals. We cite one example. Let $n \geq 0$, put $v = u^n - u^{n-1}$, and define

$$\varphi(n) = \int_0^1 \frac{\log u}{v} dv.$$

Then, for $n > 0$ [12, p. 326],

$$\varphi(n) + \varphi\left(\frac{1}{n}\right) = \frac{\pi^2}{6}, \tag{4.1}$$

which was proved and generalized by the author and R. J. Evans [22]. Although not readily apparent, (4.1) is related to a reciprocity theorem for the dilogarithm $\mathrm{Li}_2(s)$, defined for any complex number s by

$$\mathrm{Li}_2(s) = -\int_0^s \frac{\log(1-u)}{u} du, \tag{4.2}$$

where the principal branch of $\log w$ is taken. In fact, in Chapter 9 Ramanujan studied the dilogarithm, trilogarithm, and several functions akin to the dilogarithm.

One of Ramanujan's favorite and most powerful integral theorems is his "Master Theorem," which is the subject of his Quarterly Reports, written for the University of Madras, where he held a scholarship for nine months before departing for England. In this theorem Ramanujan asserts that

$$\int_0^\infty x^{n-1} \sum_{k=0}^{\infty} \frac{\varphi(k)(-x)^k}{k!} dx = \Gamma(n)\varphi(-n). \tag{4.3}$$

Of course, (4.3) needs to be properly interpreted. See Hardy's book [38, pp. 189, 190] and our book [9, pp. 298–331] for conditions of validity, many applications of (4.3) and its converse, and also for several of its variations.

5. Asymptotic Expansions and Approximations

Although Ramanujan is well known for his asymptotic formulas in number theory, in particular, for his remarkable asymptotic series for the partition function $p(n)$ in a joint paper with Hardy [39], [54, pp. 276–309], he has not been recognized for his asymptotic methods and theorems in analysis. The reason is clear; his beautiful asymptotic formulas, both general and specific, laid hidden in his notebooks for many years. Chapters 3, 13, and 15 in his second notebook [9], [10], and a long chapter in [13, Chapter 36], collecting the many results scattered in the unorganized pages of the second and third notebooks, contain most of Ramanujan's discoveries. We discuss two specific examples.

First, let $a, p > 0$. As p tends to ∞,

$$\sum_{n=0}^{\infty} \frac{(a+n)^{n-1}}{(2p+a+n)^{n+1}} \sim \frac{1}{2ap} - e^{-2p} \sum_{n=0}^{\infty} \frac{(-1)^n P_{2n}(p)}{(a+p)^{2n+1}}, \tag{5.1}$$

where $P_{2n}(p), n \geq 0$, is a polynomial in p of degree $n-1$. In particular,

$$P_0(p) = \frac{1}{2p}, \quad P_2(p) = \frac{1}{6}, \quad P_4(p) = \frac{1}{30} + \frac{p}{6}, \quad \text{and} \quad P_6(p) = \frac{1}{42} + \frac{p}{6} + \frac{5p^2}{18}.$$

For several related theorems and generalizations of (5.1), see [13] and our paper with Evans [23].

Second, as t tends to $0+$,

$$F(t) := 2\sum_{n=0}^{\infty} (-1)^n \left(\frac{1-t}{1+t}\right)^{n(n+1)} \sim 1 + t + t^2 + 2t^3 + 5t^4 + 17t^5 + \cdots. \tag{5.2}$$

More precisely, as t tends to $0+$,

$$F(t) \sim \left(\frac{1+t}{1-t}\right)^{1/4} \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{2^{2n} n!} \log^n \left(\frac{1+t}{1-t}\right). \tag{5.3}$$

Obviously, (5.2) can be easily obtained from the more complete version (5.3); the latter was not given by Ramanujan. A proof of (5.3) can be found in [13, Chapter 36]. The function $F(t)$ resembles a theta function, but it is not; it is a false theta function in the sense of L. J. Rogers [59].

6. The Gamma Function and Related Functions

The gamma function plays a ubiquitous role in Ramanujan's work, although he did not contribute any significant theorems about the gamma function itself. His integrals involving gamma functions are important in recent research on orthogonal polynomials, and his q–analogues of the gamma and beta functions arise in recent work of R. A. Askey, J. Wilson, and many others on q–orthogonal polynomials. See [10, p. 225], [11, p. 29], and [6] for more specific details and references.

In Chapters 8 and 9 of his second notebook [55], [9], Ramanujan studied several fascinating analogues of the gamma function and derived properties analogous to familiar ones for the gamma function, in particular, Gauss's multiplication formula, Stirling's formula, and Kummer's formula.

In Section 9, we discuss Ramanujan's continued fractions for products of gamma functions.

7. Hypergeometric Functions

We mentioned earlier that Ramanujan not only rediscovered most of the primary classical theorems about hypergeometric series but that he found many new results as well. First, he found many elegant product formulas for hypergeometric series. Some of these were communicated in his first two letters to Hardy, and after these letters were published with Ramanujan's *Collected Papers* [54] in 1927, W. N. Bailey and others wrote several papers on this subject. Second, Ramanujan discovered some elegant formulas for certain partial sums of hypergeometric series; some, being also found in the aforementioned letters, spawned papers in the late 1920's and early 1930's. Third, and perhaps most importantly, Ramanujan devised various asymptotic expansions for hypergeometric functions. We give one example [10, p. 195] as reformulated by Evans [31, p. 550].

Theorem. *Let $a = c+d$ with $c, d > 0$. Then, as a, c, and d tend to ∞ (equivalently, as $a/(cd) \to 0$),*

$$ {}_2F_1(a, 1; c; c/a) \sim c\left(\frac{a^a\Gamma(c)\Gamma(d)}{2\Gamma(a)c^c d^d} + B_1\frac{a}{cd} + B_2\left(\frac{a}{cd}\right)^2 + B_3\left(\frac{a}{cd}\right)^3 + \cdots\right), $$

where $B_k, k \geq 1$, is an effectively computable polynomial in $x = d/a$ of degree $2k-1$. Furthermore,

$$ \begin{aligned} B_1 &= \frac{2}{3}(x+1), \\ B_2 &= -\frac{4}{135}(x+1)(x-2)(x-\tfrac{1}{2}), \\ B_3 &= \frac{8}{2835}(x+1)(x-2)(x-\tfrac{1}{2})(x^2-x+1), \\ B_4 &= \frac{16}{8505}(x+1)(x-2)(x-\tfrac{1}{2})(x^2-x+1)^2. \end{aligned} $$

Evans [31] has given an excellent discussion of several of Ramanujan's asymptotic expansions, in particular, with emphasis on hypergeometric functions.

For a survey of hypergeometric and basic hypergeometric series emphasizing Ramanujan's contributions, see Askey's paper [5].

8. q–Series

As customary, set

$$ (a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \qquad n \geq 0, $$

and

$$ (a; q)_\infty = \lim_{n\to\infty}(a; q)_n, \qquad |q| < 1. $$

Ramanujan discovered the Rogers–Ramanujan identities in India, and although he did not prove them until sometime after he reached England, he had derived four pieces of evidence, found in the publication of the lost notebook [56, p. 358], for the validity of the first identity,

$$ \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty}. \tag{8.1} $$

In particular, as $q \to 1-$, both sides of (8.1) are asymptotically equal to

$$\exp\left(\frac{\pi^2}{15(1-q)}\right).$$

In both the third and lost notebooks, Ramanujan offers an asymptotic formula for a general class of q–series including that in (8.1).

Theorem. *Let $a > 0, |q| < 1$, and b and c be integers with $b > 0$. Let z denote the positive root of $az^{2b} + z = 1$. Then, as $q \to 1-$,*

$$\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q;q)_n} \sim \exp\left(-\frac{1}{\log q}\left(\mathrm{Li}_2(az^{2b}) + b\log^2 z\right)\right.$$
$$\left. + c\log z - \frac{1}{2}\log\left(z + 2b(1-z)\right)\right),$$

where $\mathrm{Li}_2(s)$ denotes the dilogarithm, defined by (4.2).

Two proofs are given in [12, pp. 269–284], with the better one due to R. McIntosh.

With the exception of the Rogers–Ramanujan identities, Ramanujan's ${}_1\psi_1$ summation, found as Entry 17 in Chapter 16 [11, p. 32], is undoubtedly his most famous result in the theory of q–series.

Further results on q–series may be found in [11, Chapter 16] and [12, Chapter 27].

9. Continued Fractions

The Rogers–Ramanujan continued fraction

$$(9.1) \quad R(q) := \frac{q^{1/5}}{1} \; \frac{}{+} \; \frac{q}{1} \; \frac{}{+} \; \frac{q^2}{1} \; \frac{}{+} \; \frac{q^3}{1} \; \frac{}{+} \cdots = q^{1/5}\frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \qquad |q| < 1,$$

is the only continued fraction that Ramanujan examined in print [53], [54, pp. 214–216]. However, his notebooks contain almost 200 contributions to continued fractions, and it seems to us that no person in the history of mathematics possessed the skills that Ramanujan had in determining continued fractions for various functions or finding closed form representations for continued fractions.

In his first and second letters to Hardy, Ramanujan communicated the values of $R(e^{-2\pi}), R(-e^{-\pi})$, and $R(e^{-2\pi/\sqrt{5}})$ [54, pp. xxvii, xxviii]. Other values can be found in his first and lost notebooks. For example, on page 311 of his first notebook, Ramanujan claims that

$$(9.2) \qquad R(e^{-8\pi}) = \sqrt{c^2+1} - c,$$

where

$$2c = 1 + \frac{a+b}{a-b}\sqrt{5},$$

$a = 3 + \sqrt{2} - \sqrt{5}$, and $b = (20)^{1/4}$. The first proof of (9.2) was given by H. H. Chan [16] and the author. In his first letter to Hardy, Ramanujan also asserted that

$R(e^{-\pi\sqrt{n}})$ "can be exactly found if n be any positive rational quantity." Watson [68] vaguely discussed this claim and claimed that $R(e^{-\pi\sqrt{n}})$ is an algebraic number. However, Chan, L.–C. Zhang, and the author [19] have proved that $R(e^{-\pi\sqrt{n}})$ is always a unit. K. G. Ramanathan [47], [48], [49], [50] has also extensively studied the values of $R(q)$. In his third notebook, Ramanujan also examines $R(q)$ when $|q| > 1$ and when q is a primitive nth root of unity. For $|q| > 1$, in contemporary language, Ramanujan asserted that, amazingly, the odd approximants of $R(q)$ approach $q^{1/5}/R(-1/q)$, and the even approximants approach $R(q^4)/q^{4/5}$, proved for the first time in [3]. If $q = \exp(2\pi im/n)$, where $(m,n) = 1$, Ramanujan asserted that $R(q)$ diverges if n is a multiple of 5 and converges otherwise. Moreover, when convergent, Ramanujan (somewhat imprecisely) evaluated $R(q)$. Unaware of Ramanujan's claims, I. Schur [61] first proved these assertions in 1917.

Ramanujan recorded several other continued fractions of the type (9.1) in the unorganized part of his second notebook and in his third notebook. For example,

$$\frac{1}{1}\,\underset{+}{}\,\frac{q+q^2}{1}\,\underset{+}{}\,\frac{q^4}{1}\,\underset{+}{}\,\frac{q^3+q^6}{1}\,\underset{+}{}\,\frac{q^8}{1}\,\underset{+}{}\,\cdots = \frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}, \qquad |q| < 1. \tag{9.3}$$

The first proof of (9.3) was given in 1936 by A. Selberg [62], who derived continued fractions for a large class of quotients of q–series, and so proved several results of Ramanujan, unaware that they were in Ramanujan's still unpublished notebooks. Further references to (9.3) and other continued fractions akin to (9.1) and (9.3) can be found in [3, in particular, pp. 20–23].

Ramanujan discovered continued fractions for several products of gamma functions. We cite one of several such theorems [10, p. 155, Entry 33].

Theorem. *Let x, m, and n be complex. If either m or n is an integer, or if* $\operatorname{Re} x > 0$, *then*

$$\begin{aligned}&\big\{\Gamma(\tfrac12(x+m+n+1))\Gamma(\tfrac12(x-m-n+1))\\ &\qquad -\Gamma(\tfrac12(x+m-n+1))\Gamma(\tfrac12(x-m+n+1))\big\}\div\\ &\big\{\Gamma(\tfrac12(x+m+n+1))\Gamma(\tfrac12(x-m-n+1))\\ &\qquad +\Gamma(\tfrac12(x+m-n+1))\Gamma(\tfrac12(x-m+n+1))\big\}\end{aligned}$$

$$= \frac{mn}{x}\,\underset{+}{}\,\frac{(m^2-1^2)(n^2-1^2)}{3x}\,\underset{+}{}\,\frac{(m^2-2^2)(n^2-2^2)}{5x}\,\underset{+}{}\,\cdots. \tag{9.4}$$

Many interesting continued fractions are limiting cases of these general continued fractions for products of gamma functions. For example [10, p. 145],

$$\pi = \frac{4}{1}\,\underset{+}{}\,\frac{1^2}{2}\,\underset{+}{}\,\frac{3^2}{2}\,\underset{+}{}\,\frac{5^2}{2}\,\underset{+}{}\,\cdots$$

is Lord Brouncker's continued fraction for π, and [10, p. 155]

$$\zeta(3) = 1 + \frac{1}{2\cdot 2}\,\underset{+}{}\,\frac{1^3}{1}\,\underset{+}{}\,\frac{1^3}{6\cdot 2}\,\underset{+}{}\,\frac{2^3}{1}\,\underset{+}{}\,\frac{2^3}{10\cdot 2}\,\underset{+}{}\,\cdots$$

is the continued fraction employed by R. Apéry [4] in his renowned proof of the irrationality of $\zeta(3)$.

The most celebrated continued fraction of the type (9.4) is Entry 40 of Chapter 12 in the second notebook [10, p. 163], which is a terminating continued fraction involving five parameters and which was first established by G. N. Watson [71]. It was not until 1991 that D. Masson [46] proved a nonterminating version of Entry 40 and a companion result as well.

Proofs of Ramanujan's continued fractions for products of gamma functions have been given by a variety of methods, but we probably still do not know Ramanujan's methods. In recent years, Masson [44], [45], [46] and L.–C. Zhang [75] have developed a unified approach through generalized hypergeometric series, and this may be closest to Ramanujan's ideas.

10. Theta Functions and Modular Equations

Ramanujan's approach to the theory of theta functions does not appear to have been influenced by any other writer. His general theta function

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1, \tag{10.1}$$

is an alternative formulation of the general classical theta function which includes as special cases $\vartheta_n(z,q), 1 \le n \le 4$ [73, pp. 463, 464], but for Ramanujan's purposes (10.1) is the most useful definition. Ramanujan's notation immediately yields the symmetry $f(a,b) = f(b,a)$. The three most important special cases of (10.1) for Ramanujan are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$
$$\psi(q) := \tfrac{1}{2} f(1,q) = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n-1)/2} = (q;q)_\infty = q^{-1/24}\eta(z), \tag{10.2}$$

where $q = \exp(2\pi i z), \mathrm{Im}\ z > 0, \eta(z)$ denotes the Dedekind eta–function, and the penultimate equality in (10.2) gives Euler's famous pentagonal number theorem.

Ramanujan derived an enormous number of theta function identities, many of them classical but also many of them original. We offer three examples. For $|q| < 1$,

$$\frac{\psi^3(q)}{\psi(q^3)} = 1 + 3\sum_{n=0}^{\infty}\left(\frac{q^{6n+1}}{1-q^{6n+1}} - \frac{q^{6n+5}}{1-q^{6n+5}}\right). \tag{10.3}$$

Our proof of (10.3) in [14, pp. 18–20] is preferable to that in [11, pp. 118–119, 228–229] and uses another identity of Ramanujan,

$$\sum_{n=-\infty}^{\infty} (3n+1) q^{3n^2+2n} = (q^2;q^2)_\infty (q;q^2)_\infty^2 (q^4;q^4)_\infty^2 = \psi(q^2) f^2(-q),$$

which was perhaps first proved by B. Gordon [33]. Ramanujan derived several identities in the same spirit of (10.3), and they can be found at several places in [11]. Some are classical identities connected with formulas for the numbers of representations of a positive integer n by the quadratic forms x^2+y^2, x^2+3y^2, and x^2+xy+y^2.

Some of Ramanujan's most beautiful and useful theta function identities are his 23 identities for the Dedekind eta–function [12, pp. 204–244]. We offer one example. Let

$$P = \frac{f^2(-q)}{q^{1/6}f^2(-q^3)} \qquad \text{and} \qquad Q = \frac{f^2(-q^2)}{q^{1/3}f^2(-q^6)}.$$

Then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3.$$

Applications of Ramanujan's eta–function identities can be found in [16] and [20].

At scattered places in the first and second notebooks, Ramanujan also recorded specific values for $\varphi(q)$. For example, on page 297 of his first notebook, Ramanujan wrote

$$\frac{\varphi^2(e^{-7\pi})}{\varphi^2(e^{-\pi})} = \frac{\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}}{14}(28)^{1/8}. \tag{10.4}$$

It is classical that $\varphi(e^{-\pi}) = \pi^{1/4}/\Gamma(\frac{3}{4})$ [73, p. 525], [11, p. 103], and so (10.4) gives an exact evaluation of $\varphi(e^{-7\pi})$. Proofs of (10.4) and Ramanujan's other values for $\varphi(q)$ can be found in our paper with Chan [17]. Moreover, Chan, Zhang and the author have also shown that $\varphi(e^{-n\pi})/\varphi(e^{-\pi})$ is algebraic for every positive integer n [21].

Let

$$K := K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{10.5}$$

denote the complete elliptic integral of the first kind, where $k, 0 < k < 1$, is called the *modulus*. It is easy to show that

$$K(k) = \frac{\pi}{2}\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;k^2). \tag{10.6}$$

One of the classical and most important theorems in the theory of elliptic functions is the "inversion formula" (10.8) below. As usual in the theory of elliptic functions, set

$$q = \exp\left(-\pi\frac{K'}{K}\right) = \exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-k^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)}\right), \tag{10.7}$$

where K is defined by (10.5) and $K' = K(k')$, where $k' = \sqrt{1-k^2}$ is called the *complementary modulus*. Then

$$\varphi^2(q) = {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;k^2) =: z. \tag{10.8}$$

For Ramanujan's proof of (10.8), see [11, pp. 98–101] or [14, pp. 24–27]. Formula (10.8) and several elementary identities for theta functions can be utilized to evaluate theta functions in terms of the parameters q, k, and z. In Entries 10–12 of Chapter 17 in his second notebook [11, pp. 122–124], Ramanujan offers a "catalogue" of these evaluations. For example,

$$\varphi(-q^2) = \sqrt{z}(1-k^2)^{1/8} \qquad \text{and} \qquad \psi(q^2) = \tfrac{1}{2}\sqrt{z}(k^2/q)^{1/4}.$$

This catalogue serves as the foundation for Ramanujan's modular equations.

There are several definitions of a modular equation. The one frequently given in the theory of modular forms [58, p. 118] is slightly different from that used by Ramanujan, and we give the latter definition now. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', ℓ, and ℓ', respectively. Suppose that, for some fixed positive integer n,

$$\frac{L'}{L} = n\frac{K'}{K}. \tag{10.9}$$

Then a modular equation of degree n is a relation between the moduli k and ℓ which is implied by (10.9). The multiplier m, which often appears in modular equations, is defined by

$$m = \frac{\varphi^2(q)}{\varphi^2(q^n)}. \tag{10.10}$$

In view of (10.7), if we set $q' = \exp(-\pi L'/L)$, then (10.9) is equivalent to the relation $q^n = q'$. It is well known that k and ℓ can be expressed in terms of theta functions [11, p. 102], and so a modular equation of degree n can also be viewed as an identity among theta functions at the arguments q and q^n. Ramanujan did not employ the classical notation; he set $\alpha = k^2$ and $\beta = \ell^2$. Ramanujan also derived modular equations involving four moduli. These "mixed" modular equations, or modular equations of composite degree, arise from a triple of relations of the form (10.9). For a more precise definition of "mixed" modular equation, see [11, p. 325] or [14, p. 30].

In one sense, the theory of modular equations began with the transformations of Gauss and Landen giving rise to modular equations of degree 2 [14, pp. 30, 31]. However, the history of the subject is usually considered to have begun in 1825 with Legendre's modular equation of degree 3 [41]. In the next 100 years, many modular equations were derived by E. Fielder, R. Fricke, A. G. Greenhill, C. Guetzlaff, M. Hanna, C. G. J. Jacobi, F. Klein, R. Russell, L. Schläfli, H. Schröter, H. Weber, and others. However, in his notebooks, Ramanujan perhaps recorded more modular equations than those of his predecessors combined. Chapters 19–21 in his second notebook are almost completely devoted to modular equations. Dozens of others can be found scattered among the unorganized portions of both his first and second notebooks. See [11, Chapters 19–21], [12, Chapter 25], and [13, Chapter 36] for proofs of all of Ramanujan's modular equations in his notebooks.

We next cite a few examples. Legendre's modular equation of degree 3, also discovered by Ramanujan [11, p. 230, Entry 5(ii)], is given by

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1.$$

A modular equation in $\alpha\beta$ and $(1-\alpha)(1-\beta)$ is often called an *irrational modular equation.* Such modular equations are often the simplest and very useful, and they have been derived for several values of n, but as n increases, they become exceedingly more complicated.

Another useful type of modular equation is the *Schläfli modular equation.* Let

$$P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \qquad \text{and} \qquad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}.$$

Then

$$Q + \frac{1}{Q} + 2\sqrt{2}\left(P - \frac{1}{P}\right) = 0 \tag{10.11}$$

is a modular equation of degree 3, first established by L. Schläfli [60] in 1870 and rediscovered by Ramanujan [11, p. 231, Entry 5(xii)]. Such equations are very important in the calculation of class invariants, discussed in Section 12.

Our third example is a modular equation of degree 17 [11, p. 397, Entry 12(iii)]:

$$\begin{aligned} m = &\left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \\ &- 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}\left(1 + \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8}\right). \end{aligned} \tag{10.12}$$

Many methods have been developed to prove modular equations, but a method that may be applicable for one type of equation or for one degree may be inapplicable for another type of equation or degree. Generally, a modular equation is expressed, via the aforementioned catalogue of "evaluations," as a theta function identity, and the task is then to prove the theta function identity. We do not know Ramanujan's methods, but he apparently used basic elementary properties of theta functions, the Jacobi triple product identity, the quintuple product identity, transformation formulas for theta functions, his ${}_1\psi_1$ summation formula, and Lambert series. Almost certainly, he used Schröter's formulas for theta functions [11, pp. 65–76]. He also may have used parametrizations of $\alpha, \beta, 1-\alpha$, and $1-\beta$ in conjunction with previously derived modular equations to find new modular equations. However, for many of Ramanujan's modular equations, (10.12), for example, we have been unable to use tools that might have been known to Ramanujan and so have had to resort to the theory of modular forms. Although this is a very powerful tool, it is pedagogically unsatisfactory, because the modular equation needs to be known in advance to effect a proof.

Andrews [2] and L.–C. Shen [63], [64], [65], [66] have recently found new proofs of some of Ramanujan's modular equations of degrees 3 and 5.

Ramanathan's paper [51] and our lecture notes [14] provide introductions to much of Ramanujan's work on modular equations.

11. Ramanujan's Theories of Elliptic Functions to Alternative Bases

In his famous paper on modular equations and approximations to π [52], [54, pp. 23–39], Ramanujan offers several series representations for $1/\pi$, for example (see also [26]),

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(6n+1)(\frac{1}{2})_n^3}{(n!)^3 4^n},$$

where $(a)_n := a(a+1)\cdots(a+n-1), n \geq 0$. He asserts [54, p. 37] that some of his formulas are derived from " ... corresponding theories in which q is replaced by one or other of the functions ... ." In the notation (10.6) and (10.7), Ramanujan replaced q by

$$q_r := \exp\left(-\pi \csc(\pi/r) \frac{{}_2F_1(\frac{1}{r}, \frac{1-r}{r}; 1; 1-k^2)}{{}_2F_1(\frac{1}{r}, \frac{1-r}{r}; 1; k^2)}\right), \tag{11.1}$$

where $r = 3, 4$, or 6. Note that $q_2 = q$, as given by (10.7). The first six pages in the unorganized part of the second notebook [55, pp. 257–262] are devoted to these theories. Except for a few entries verified by K. Venkatachaliengar [67], no one had examined this material until S. Bhargava, F. G. Garvan, and the author [15] completed proofs for all these entries. The theories for $r = 4$ and 6 hinge chiefly on the classical theory when $r = 2$. The theory for $r = 3$ is more interesting and difficult. Fortunately, shortly after the authors began their efforts, J. M. and P. B. Borwein [27] (see also [28]) discovered a cubic analogue of a famous identity of Jacobi [11, p. 40, Entry 25(vii)] for their cubic theta functions. This was necessary for us to obtain an analogue of (10.8). We now state the Borweins' cubic theta functions, their cubic theta function identity, and the analogue of (10.8).

Let

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$
$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2},$$

and

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2},$$

where $\omega = \exp(2\pi i/3)$. Then [27], [28]

$$a^3(q) = b^3(q) + c^3(q). \tag{11.2}$$

The analogue of (10.8) is given by [15, Theorem 2.10]

$$a(q_3) = {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; k^2\right) =: z_3. \tag{11.3}$$

Besides (11.2), a key ingredient in proving (11.3) is the cubic transformation formula

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\left(\frac{1-x}{1+2x}\right)^3\right) = (1+2x)\ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x^3\right), \tag{11.4}$$

where $0 \leq x < 1$. This transformation is found in Ramanujan's notebooks and was rediscovered by the Borweins [27].

The identity (11.2) is not found in the notebooks, but a short proof of it can be constructed by using material from the notebooks [15, Lemma 2.1, Theorem 2.2], [14, Theorem 13.3]. It appears inconceivable to us that Ramanujan could have derived (11.3) without knowing (11.2) or an equivalent result.

Ramanujan, in fact, offered several new transformation formulas for hypergeometric functions on the aforementioned six pages. Moreover, we have used computer algebra to derive generalizations of these transformations [15]. See Garvan's paper [32] to learn how MAPLE was utilized in these discoveries.

The problem of inverting elliptic integrals is central in the theory of elliptic functions. The following inversion formula of Ramanujan is an analogue of one of these inversion formulas.

Theorem. *Let q_3 and z_3 be given by (11.1) and (11.3), respectively. For $0 \leq \varphi \leq \pi/2$ and $0 \leq x < 1$, define $\theta = \theta(\varphi)$ by*

$$\theta z_3 = \int_0^{\varphi} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x \sin^2 t\right) dt. \tag{11.5}$$

Then, for $0 \leq \theta \leq \pi/2$,

$$\varphi = \theta + 3 \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n(1 + 2\cosh(ny))},$$

where $q_3 =: \exp(-y)$.

Recalling that [10, p. 99, Entry 35(iii)]

$${}_2F_1\left(\frac{1}{2} + a, \frac{1}{2} - a; \frac{1}{2}; x^2\right) = (1 - x^2)^{-1/2} \cos(2a \arcsin x),$$

where a is any complex number, we see that, if $a = 0$ and x^2 is replaced by $x \sin^2 t$, then (11.5) is a natural analogue of the inversion formula for the incomplete elliptic integral of the first kind.

Ramanujan recorded several modular equations in the alternative theories. For example, if $r = 3$, then [15, Theorem 7.5]

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} + 3\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1$$

is a modular equation of degree 3.

12. Class Invariants

Let $\chi(q) = (-q; q^2)_\infty$. If $q = \exp(-\pi\sqrt{n})$, where n is any positive rational number, then Ramanujan's two class invariants G_n and g_n are defined by

$$G_n := 2^{-1/4} q^{-1/24} \chi(q) \qquad \text{and} \qquad g_n := 2^{-1/4} q^{-1/24} \chi(-q).$$

In the notation of Weber [72], $G_n =: 2^{-1/4}\mathfrak{f}(\sqrt{-n})$ and $g_n =: 2^{-1/4}\mathfrak{f}_1(\sqrt{-n})$. It is well known that G_n and g_n are algebraic; for example, see Weber's book [72] or Cox's book [30, p. 214, Theorem 10.23; p. 257, Theorem 12.17].

At scattered places in his first notebook, Ramanujan recorded the values for 107 class invariants. On pages 294–299 in his second notebook, Ramanujan gave a table of values for 77 class invariants, three of which are not found in the first notebook. After coming to England, Ramanujan became aware of Weber's work, and so his table of 46 class invariants in [52] does not contain any that are found in Weber's book [72]. Watson wrote eight papers on class invariants, and, in particular, in [69] and [70], he established 24 of Ramanujan's class invariants. After Watson's work, 18 of Ramanujan's class invariants remained to be verified, but these have now been proved in two papers [18], [20] with H. H. Chan and L.–C. Zhang. We briefly describe this work.

In [18], five of Ramanujan's invariants are proved by appealing to two new theorems that relate G_{9n} with G_n, and g_{9n} with g_n, respectively. The modular equation (10.11) is the primary result used in the proofs. Thus, for example, using the known value for G_{17}, we showed that

$$G_{153} = \left(\sqrt{\frac{5+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-3}{8}}\right)^2 \left(\sqrt{\frac{37+9\sqrt{17}}{4}} + \sqrt{\frac{33+9\sqrt{17}}{4}}\right)^{1/3}.$$

The remaining thirteen values of G_n were more difficult to prove. In each case, the class number of $\mathbb{Q}(\sqrt{-n})$ equals 8, and there are two classes per genus. Three methods were employed. The first utilizes Ramanujan's modular equations, and so is in the spirit of Ramanujan. For example, to prove that

$$G_{65} = \left(\frac{\sqrt{13}+3}{2}\right)^{1/4} \left(\frac{\sqrt{5}+1}{2}\right)^{1/4} \left(\sqrt{\frac{9+\sqrt{65}}{8}} + \sqrt{\frac{1+\sqrt{65}}{8}}\right)^{1/2},$$

a modular equation of degree 5, similar to (10.11), and one of Ramanujan's eta–function identities are employed. However, we were able to use modular equations to prove only six of the thirteen invariants. The second method depends upon Kronecker's limit formula and can be used to establish all thirteen class invariants, but these ideas were not known to Ramanujan. The third employs class field theory to make Watson's "empirical process" in [69] rigorous. Although Watson was confident that Ramanujan employed this "empirical process," we disagree. In summary, Ramanujan's proofs for several of his class invariants remain elusive.

Many mathematicians have either individually or in collaboration with the author proved many of Ramanujan's difficult claims in the notebooks. The following list is incomplete, and we apologize to those whose names we have omitted. Each has proved theorems of Ramanujan that would remain unproved without their efforts. We offer our sincere thanks to: George E. Andrews, Richard A. Askey, Gennady Bachman, S. Bhargava, Anthony J. Biagioli, David Bradley, Heng Huat Chan, Henri Cohen, Ronald J. Evans, Frank G. Garvan, James L. Hafner, Lisa Lorentzen, Kenneth S. Williams, Don Zagier, and Liang–Cheng Zhang.

References

1. G. E. Andrews, *Ramanujan's "lost" notebook III. The Rogers–Ramanujan continued fraction*, Adv. Math. **41** (1981).
2. G. E. Andrews, *Modular equations and Ramanujan's Chapter 16, Entry 29*, Proc. Indian Acad. Sci. (Math. Sci.) **104** (1994), 225–235.
3. G. E. Andrews, B. C. Berndt, L. Jacobsen, R. L. Lamphere, *The continued fractions found in the unorganized portions of Ramanujan's notebooks*, Mem. Amer. Math. Soc., No. 477 **99** (1992).
4. R. Apéry, *Interpolation de fractions continues et irrationalite de certaines constantes*, Bull. Section des Sci., Tome III (1981), Bibliothéque Nationale, Paris, 37–63.
5. R. A. Askey, *Ramanujan and hypergeometric and basic hypergeometric series*, Russian Math. Surveys **45** (1990), 37–86; *Ramanujan International Symposium on Analysis, Pune 26–28 December 1987*, N. K. Thakare, ed., Macmillan India, New Delhi, 1989, pp. 1–83.
6. R. A. Askey, *Beta integrals and q–extensions*, Proceedings of the Ramanujan Centennial International Conference, 15–18 December 1987, Annamalainagar, Ramanujan Mathematical Society, Madras, 1988, pp. 85–102.
7. B. C. Berndt, *Modular transformations and generalizations of several formulae of Ramanujan*, Rocky Mt. J. Math. **7** (1977), 147–189.
8. B. C. Berndt, *Analytic Eisenstein series, theta–functions, and series relations in the spirit of Ramanujan*, J. Reine Angew. Math. **303/304** (1978), 332–365.
9. B. C. Berndt, *Ramanujan's Notebooks, Part I*, Springer–Verlag, New York, 1985.
10. B. C. Berndt, *Ramanujan's Notebooks, Part II*, Springer–Verlag, New York, 1989.
11. B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer–Verlag, New York, 1991.
12. B. C. Berndt, *Ramanujan's Notebooks, Part IV*, Springer–Verlag, New York, 1994.
13. B. C. Berndt, *Ramanujan's Notebooks, Part V*, Springer–Verlag, New York, 1998.
14. B. C. Berndt, *Ramanujan's theory of theta–functions*, Theta Functions from the Classical to the Modern (M. Ram Murty, ed.), Centre de Recherches Math., CRM Proc. and Lecture Notes, vol. 1, pp. 1–63.
15. B. C. Berndt, S. Bhargava, and F. Garvan, *Ramanujan' theories of elliptic functions to alternative bases*, Trans. Amer. Math. Soc. **347** (1995), 4163–4244.
16. B. C. Berndt and H. H. Chan, *Some values for the Rogers–Ramanujan continued fraction*, Canad. J. Math. **47** (1995), 897–914.
17. B. C. Berndt and H. H. Chan, *Ramanujan's explicit values for the classical theta–function*, Mathematika **42** (1995), 278–294.
18. B. C. Berndt, H. H. Chan, and L.–C. Zhang, *Ramanujan's class invariants and cubic continued fraction*, Acta Arith. **73** (1995), 67–85.
19. B. C. Berndt, H. H. Chan, and L.–C. Zhang, *Explicit evaluations of the Rogers–Ramanujan continued fraction*, J. Reine Angew. Math. **480** (1996), 141–159.
20. B. C. Berndt, H. H. Chan, and L.–C. Zhang, *Ramanujan's class invariants, Kronecker's limit formula, and modular equations*, Trans. Amer. Math. Soc. **349** (1997), 2125–2173.
21. B. C. Berndt, H. H. Chan, and L.–C. Zhang, *Ramanujan's remarkable product of theta–functions*, Proc. Edinburgh Math. Soc. **40** (1997), 583–612.
22. B. C. Berndt and R. J. Evans, *An integral functional equation of Ramanujan related to the dilogarithm*, Number Theory (R. A. Mollin, ed.), de Gruyter, Berlin, 1990, pp. 1–5.
23. B. C. Berndt and R. J. Evans, *Asymptotic expansion of a series of Ramanujan*, Proc. Edinburgh Math. Soc. **35** (1992), 189–199.
24. B. C. Berndt and J. L. Hafner, *Two remarkable doubly exponential series transformations of Ramanujan*, Proc. Indian Acad. Sci. (Math. Sci.) **104** (1994), 245–252.
25. B. C. Berndt and R. A. Rankin, *Ramanujan: Letters and Commentary*, American Mathematical Society, Providence, RI, 1995; jointly published by the London Mathematical Society, London, 1995.
26. J. M. and P. B. Borwein, *Pi and the AGM*, Wiley, New York, 1987.
27. J. M. and P. B. Borwein, *A cubic counterpart of Jacobi's identity and the AGM*, Trans. Amer. Math. Soc. **323** (1991), 691–701.
28. J. M. Borwein, P. B. Borwein, and F. G. Garvan, *Some cubic modular identities of Ramanujan*, Trans. Amer. Math. Soc. **343** (1994), 35–47.
29. G. S. Carr, *Formulas and Theorems in Pure Mathematics, 2nd ed.*, Chelsea, New York, 1970.
30. D. A. Cox, *Primes of the Form $x^2 + ny^2$*, Wiley, New York, 1989.

31. R. J. Evans, *Ramanujan's second notebook: Asymptotic expansions for hypergeometric series and related functions*, Ramanujan Revisited (G. E. Andrews, R. A. Askey, B. C. Berndt, K. G. Ramanathan, and R. A. Rankin, eds.), Academic Press, Boston, 1988, pp. 537–560.
32. F. G. Garvan, *Ramanujan's theories of elliptic functions to alternative bases - a symbolic excursion*, J. Symbolic Comp. **20** (1995), 517–536.
33. B. Gordon, *Some identities in combinatorial analysis*, Quart. J. Math. (Oxford) (2) **12** (1961), 285–290.
34. E. Grosswald, *Die Werte der Riemannschen Zeta–funktion an ungeraden Argumentstellen*, Nachr. Akad. Wiss. Göttingen (1970), 9–13.
35. E. Grosswald, *Comments on some formulae of Ramanujan*, Acta Arith. **21** (1972), 25–34.
36. G. H. Hardy, *A chapter from Ramanujan's note–book*, Proc. Cambridge Philos. Soc. (2) **21** (1923), 492–503.
37. G. H. Hardy, *Collected Papers, Vol. 4*, Clarendon Press, Oxford, 1969.
38. G. H. Hardy, *Ramanujan*, Cambridge University Press, Cambridge, 1940; reprinted by Chelsea, New York, 1978; reprinted by the American Mathematical Society, Providence, RI, 1999.
39. G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (2) **17** (1918), 75–118.
40. R. Kanigel, *The Man Who Knew Infinity*, Charles Scribner's, New York, 1991.
41. A. M. Legendre, *Traité des Fonctions Elliptiques I*, Huzard–Courcier, Paris, 1825.
42. A. M. Legendre, *Thórie des Nombres, t. 1, 4th ed.*, Librairie Scientifique, Hermann, Paris, 1830.
43. M. Lerch, *Sur la fonction $\zeta(s)$ pour valeurs impaires d'argument*, J. Sci. Math. Astron., pub. pelo Dr. F. Gomes Teixeira, Coimbra **14** (1901), 65–69.
44. D. Masson, *Some continued fractions of Ramanujan and Meixner–Pollaczek polynomials*, Canad. Math. Bull. **32** (1989), 177–181.
45. D. Masson, *Wilson polynomials and some continued fractions of Ramanujan*, Rocky Mt. J. Math. **21** (1991), 489-499.
46. D. Masson, *A generalization of Ramanujan's best theorem on continued fractions*, C. R. Math. Rep. Acad. Sci. Canada **13** (1991), 167–172.
47. K. G. Ramanathan, *On Ramanujan's continued fraction*, Acta Arith. **43** (1984), 209–226.
48. K. G. Ramanathan, *On the Rogers–Ramanujan continued fraction*, Proc. Indian Acad. Sci. (Math. Sci.) **93** (1984), 67–77.
49. K. G. Ramanathan, *Ramanujan's continued fraction*, Indian J. Pure Appl. Math. **16** (1985), 695–724.
50. K. G. Ramanathan, *Some applications of Kronecker's limit formula*, J. Indian Math. Soc. **52** (1987), 71–89.
51. K. G. Ramanathan, *Ramanujan's modular equations*, Acta Arith. **53** (1990), 403–420.
52. S. Ramanujan, *Modular equations and approximations to π*, Quart. J. Math. (Oxford) **45** (1914), 350–372.
53. S. Ramanujan, *Proof of certain identities in combinatory analysis*, Proc. Cambridge Philos. Soc. **19** (1919), 214–216.
54. S. Ramanujan, *Collected Papers*, University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.
55. S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.
56. S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
57. R. A. Rankin, *George Neville Watson*, J. London Math. Soc. **41** (1966), 551–565.
58. R. A. Rankin, *Modular Forms and Functions*, Cambridge University Press, Cambridge, 1977.
59. L. J. Rogers, *On two theorems of combinatory analysis and some allied identities*, Proc. London Math. Soc. **16** (1916), 315–336.
60. L. Schläfli, *Beweis der Hermiteschen Verwandlungstafeln für die elliptischen modul Funktionen*, J. Reine Angew. Math. **72** (1870), 360–369.
61. I. J. Schur, *Ein Beiträg zur additiven Zahlentheorie und zur Theorie der Kettenbrüche*, S.–B. Preuss. Akad. Wiss. Phys.–Math. Kl. (1917), 302–321.
62. A. Selberg, *Über einige arithmetische Identitäten*, Avh. Norske Vid.–Akad. Oslo I. Mat.–Naturv. K., No. 8 (1936), 3–23.
63. L.–C. Shen, *On some cubic modular identities*, Proc. Amer. Math. Soc. **119** (1993), 203–208.

64. L.–C. Shen, *On the modular equations of degree 3*, Proc. Amer. Math. Soc. **122** (1994), 1101–1114.
65. L.–C. Shen, *On the additive formulae of the theta functions and a collection of Lambert series pertaining to the modular equations of degree 5*, Trans. Amer. Math. Soc. **345** (1994), 323–345.
66. L.–C. Shen, *On some modular equations of degree 5*, Proc. Amer. Math. Soc. **123** (1995), 1521–1526.
67. K. Venkatachaliengar, *Development of Elliptic Functions According to Ramanujan*, Tech. Rep. 2, Madurai Kamaraj University, Madurai, 1988.
68. G. N. Watson, *Theorems stated by Ramanujan (VII): theorems on continued fractions*, J. London Math. Soc. **4** (1929), 39–48.
69. G. N. Watson, *Some singular moduli (I)*, Quart. J. Math. **3** (1932), 81–98.
70. G. N. Watson, *Some singular moduli (II)*, Quart. J. Math. **3** (1932), 189–212.
71. G. N. Watson, *Ramanujan's continued fraction*, Proc. Cambridge Philos. Soc. **31** (1935), 7-17.
72. H. Weber, *Lehrbuch der Algebra, dritter Band*, Chelsea, New York, 1961.
73. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis, 4th ed.*, University Press, Cambridge, 1966.
74. D. A. B. Young, *Ramanujan's illness*, Notes Rec. Royal Soc. London **48** (1994), 107–119.
75. L.–C. Zhang, *Ramanujan's continued fractions for products of gamma functions*, J. Math. Anal. Applics. **174** (1993), 22–52.